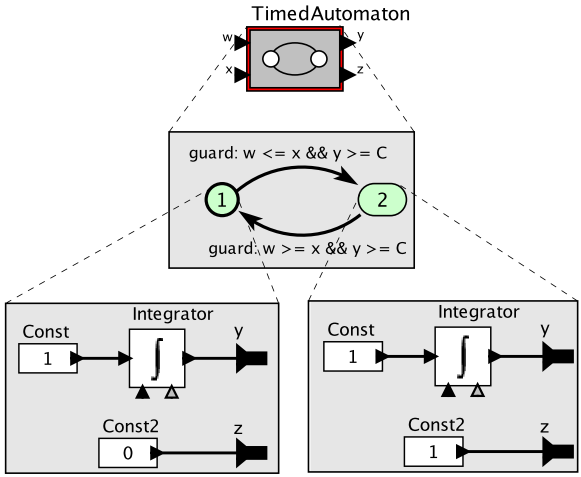

Figure 7.1: A timed automaton.

2. This problem explores scenarios in which the ideal solver semantics is actually practical and realizable. Consider the model shown in Figure 7.1. This is a simple case of a timed automaton (Alur and Dill, 1994). Assume $C > 0$ is a given real constant, and assume that the mode transitions are both reset transitions, so the state of the integrators is reset to zero on each transition. This model could provide an implementation for a thermostat, with the following associations for the signals:

   - w is the current temperature;
   - x is the current set point (the target temperature);
   - y is the time elapsed since entering a mode; and
   - z is the control signal turning a heater on (1) or off (0).

   Let the start time of the model be $t_0 = 0$ and the time of the n^{th} mode transition be given by t_n. In each state $k \in \{1, 2\}$, the system needs to solve an initial value problem of the form

   $$\dot{y}(t) = f_k(y(t)), \quad y(t_n) = 0$$

   for $t \geq t_n$.

   (a) Give the functions f_1 and f_2.

   (b) Show that f_1 and f_2 are Lipschitz continuous.

(c) The initial-value problem has an analytical solution, meaning that there is a closed-form expression for $y(t)$ for all $t \geq t_k$. Find that solution.

(d) Let $(X, \leq)$ be a total order. A subset $T \subseteq X$ is **progressing** if for all $x \in X$, there exists a $t \in T$ such that $x \leq t$. Let T be the set of times t_n at which a transition is taken. Show that either T is finite or T is progressing.

(e) Explain why the ideal-solver semantics is actually implementable in this case. You may assume that arithmetic on time values suffers no quantization errors when addition or subtraction is performed, and you may assume that the times at which $w < x$ or $w > x$ become true are representable exactly in the number system used by your computer.

(f) Consider the model of Figure 7.1 in a feedback loop with a physical plant representing the cooling and heating of the room. Suppose that the temperature of the room is given by

$$w(t) = w(0) + \int_0^t (hz(\tau) - r(1 - z(\tau))d\tau,$$

where h is constant representing the rate of heating when the heater is on (when $z(t) = 1$), r is the rate of cooling when the heater is off (when $z(t) = 0$), and $w(0)$ is the initial temperature of the room. Comment about the quality of this thermostat. In particular, suppose $C = 20$ seconds, $w(0) = 22$ degrees centigrade, $h = 0.2$ degrees per second, and $r = 0.1$ degrees per second. How often will the heater turn on an off? What are the maximum and minimum temperatures reached? Can you suggest a better thermostat design?