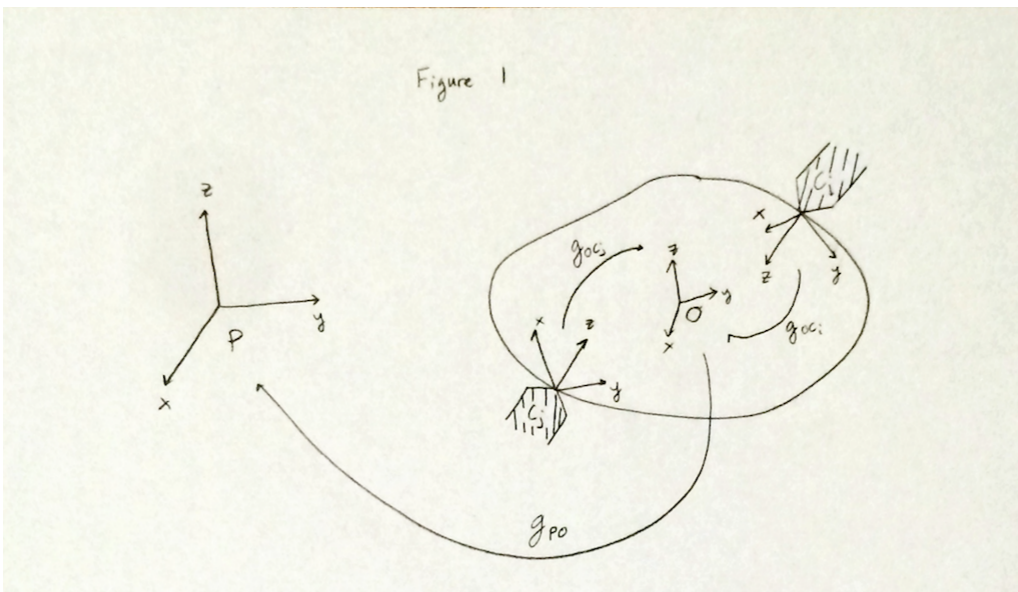
Florian's Notes

March 2016

Wrenches at O



First Task: Understand overall wrench contribution.

Recall:

a.

$$F_b = \begin{bmatrix} f_b \\ \tau_b \end{bmatrix} \begin{matrix} \leftarrow & \text{force} \in \mathbb{R}^3 \\ \leftarrow & \text{moment} \in \mathbb{R}^3 \end{matrix}$$

b.

$$F_a = Ad^T_{g_{ab}} F_b$$

Applying this to the figure, we have: $F_o = Ad^T_{g_{c_i o}} F_{c_i} = Ad^T_{g^{-1}_{oc_i}} F_{c_i}$

Once all the generalized forces are expressed in the O-frame, we can sum the forces from the contact frames $C_1, ..., C_n$

$$F_o = \sum_{i=1}^{n} Ad^T_{g_{c_i o}} F_{c_i}$$

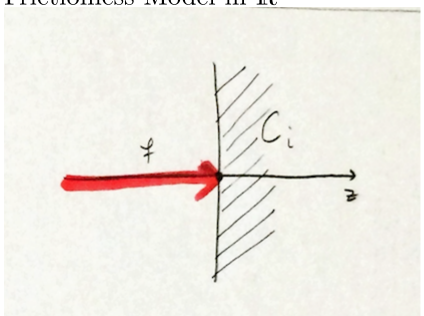
to obtain the total wrench acting at O.

Note

$$\begin{aligned}
g_{oc_i}^{-1} &= \begin{bmatrix} R_{oc_i} & p_{oc_i} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R_{oc_i}^T & -R_{oc_i}^T p_{oc_i} \\ 0 & 1 \end{bmatrix} \\
Ad^T_{g_{oc_i}^{-1}} F_{c_i} &= \begin{bmatrix} R_{oc_i}^T & -R_{oc_i}^T p_{oc_i} R_{oc_i}^T \\ 0 & R_{oc_i}^T \end{bmatrix}^T = \begin{bmatrix} R_{oc_i} & 0 \\ -\hat{p_{oc_i}} R_{oc_i} & R_{oc_i} \end{bmatrix} \\
R^{\hat{T}} p R^T &= R^T \hat{p}, \text{ from chapter 2 of MLS}
\end{aligned}$$

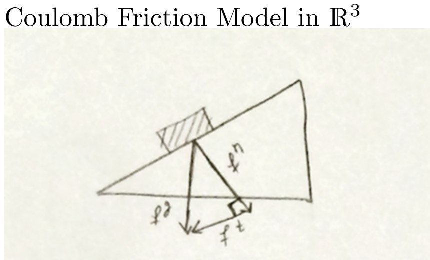
What wrenches can we apply?

a. Frictionless Model in $\mathbb{R}^3$

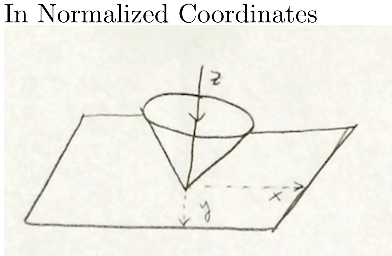


$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} f_{c_i} \in FC \qquad FC_{c_i} = \{f | f > 0\}$$

b. Coulomb Friction Model in $\mathbb{R}^3$

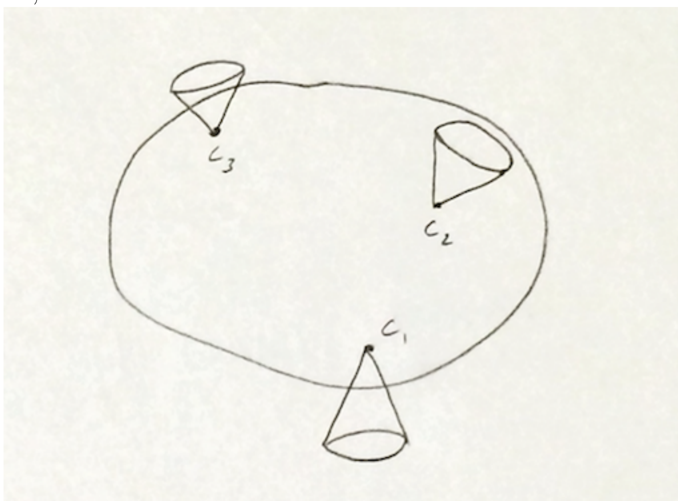


Slipping starts when $|f^t| > \mu f^n$, so we want $|f^t| \le \mu f^n$ and $\alpha = \tan^{-1} \mu$
In Normalized Coordinates



$$f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \qquad f^n = f_z, \ |f^t| = | \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} | = \sqrt{f_x^2 + f_y^2}$$

So,



We can apply wrenches in the form

$$F_{c_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} f_{c_i} \qquad f_{c_i} \in FC_{c_i} \subseteq \mathbb{R}^3$$

$$FC_{c_i} = \{f \in \mathbb{R}^3 | \sqrt{f_1^2 + f_2^2} \leq \mu f_3, f_3 > 0\}$$

Examples of Friction Coefficients

steel on steel: 0.58

wood on metal: 0.2 - 0.6

rubber on solid: 1-4

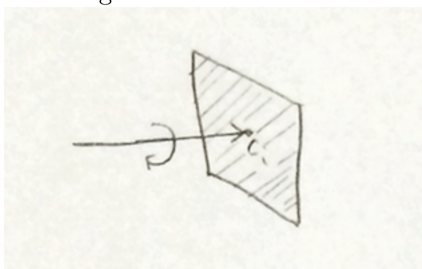
Caveats

How do we estimate μ?

Do we know the material properties (to the first order)?

How do we account for environmental factors such as dust and moisture?

c. Soft Finger Contact: Additional ability to apply torque along normal



Wrenches take the form

$$F_{c_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} f_{c_i} \qquad f_{c_i} \in FC_{c_i} \subseteq \mathbb{R}^4$$

$$FC_{c_i} = \{f \in \mathbb{R}^3 | \sqrt{f_1^2 + f_2^2} \leq \mu f_3, f_3 > 0, |f_4| \leq \gamma f_3\}$$

where γ is the torsional friction coefficient.

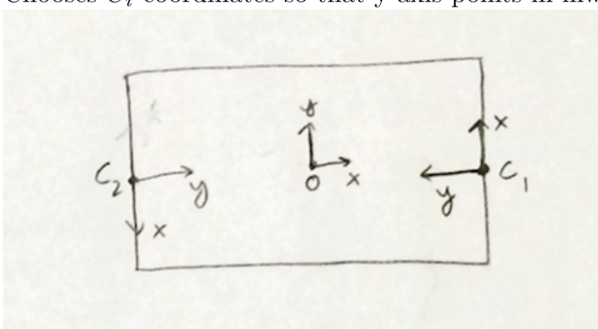
The Planar Case

Recall that planar wrenches live $\mathbb{R}^3$ and

$$\begin{bmatrix} f_o \in \mathbb{R}^2 \\ \tau_o \in \mathbb{R} \end{bmatrix} = Ad^T_{g^{-1}_{oc_i}} \begin{bmatrix} f_{c_i} \\ \tau_{c_i} \end{bmatrix}$$

$$= \begin{bmatrix} R_{c_i} & 0 \\ (-p_y p_x) R_{c_i} & 1 \end{bmatrix} \begin{bmatrix} f_{c_i} \\ \tau_{c_i} \end{bmatrix}$$

Chooses C_i coordinates so that y-axis points in inwards normal direction.



1. Frictionless point contact:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} f_{c_i} \qquad f_{c_i} > 0 \qquad FC_{c_i} = \{f_{c_i} \in \mathbb{R} : f_{c_i} \geq\}$$

2. Point contact with friction:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} f_{c_i} \qquad f_{c_i} > 0 \qquad FC_{c_i} = \{f_{c_i} \in \mathbb{R}^3 || f_1| < \mu f_2, f_2 \geq 0\}$$

Let us denote by $p = 6$ (in 3D) and $p = 3$ (in 2D) the dimension of the wrench space.

The Grasp Map

Observe: We madel a contact using a *wrench basis* $B_{c_i} \in \mathbb{R}^{p\times m_i}$ and a friction cone FC_{c_i}. The allowable wrench forces are given by

$$F_{c_i} = B_{c_i} f_{c_i} \qquad f_{c_i} \in FC_{c_i}$$

Recall

$$F_o = Ad^T_{g^{-1}_{oc_i}} F_{c_i} = \begin{bmatrix} R_{oc_i} & 0 \\ -\hat{p_{oc_i}} R_{oc_i} & R_{oc_i} \end{bmatrix} B_{c_i} f_{c_i}, \quad f_{c_i} \in FC_{c_i}$$

Define $G_i := Ad^T_{g_{c_i o}} B_{c_i} \in \mathbb{R}^{p\times m_i}$, then the total wrench for k contacts is

$$F_o = \sum_{i=1}^{k} G_i f_{c_i} = \begin{bmatrix} G_1 & . & . & . & G_k \end{bmatrix} \begin{bmatrix} f_{c_1} \\ . \\ . \\ . \\ f_{c_k} \end{bmatrix}$$

and we call

$$G = \begin{bmatrix} G_1 & . & . & . & G_k \end{bmatrix} = \begin{bmatrix} Ad^T_{g^{-1}_{oc_1}} B_{c_1} & . & . & . & Ad^T_{g^{-1}_{oc_k}} B_{c_k} \end{bmatrix} \in \mathbb{R}^{p\times \sum m_i}$$

The *Grasp Map* and the object wrench can be written as

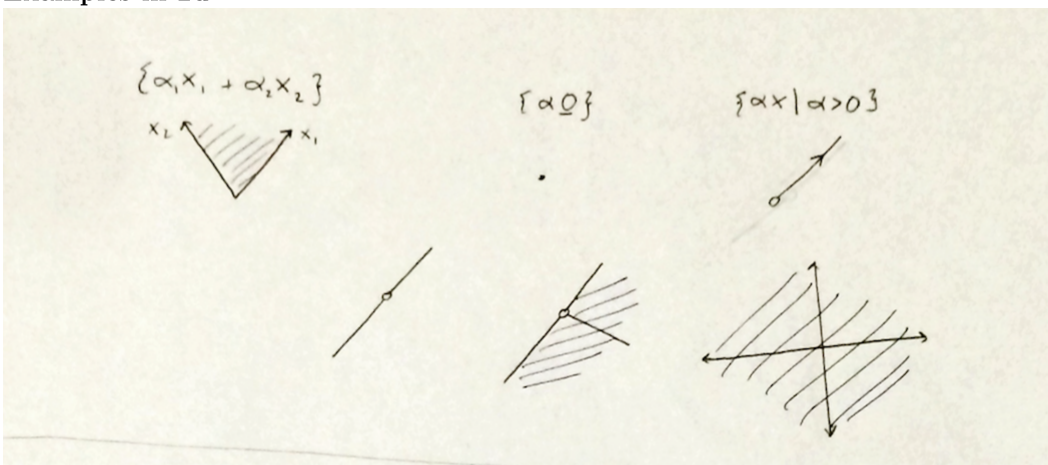
$$F_o = G f_c, \qquad f_c \in FC$$

where $f_c = (f_{c_1}, ..., f_{c_k}) \in \mathbb{R}^m$, $FC = FC_{c_1} \times ... \times FC_{c_k}$, and $m = \sum_{i=1}^{k} m_i$

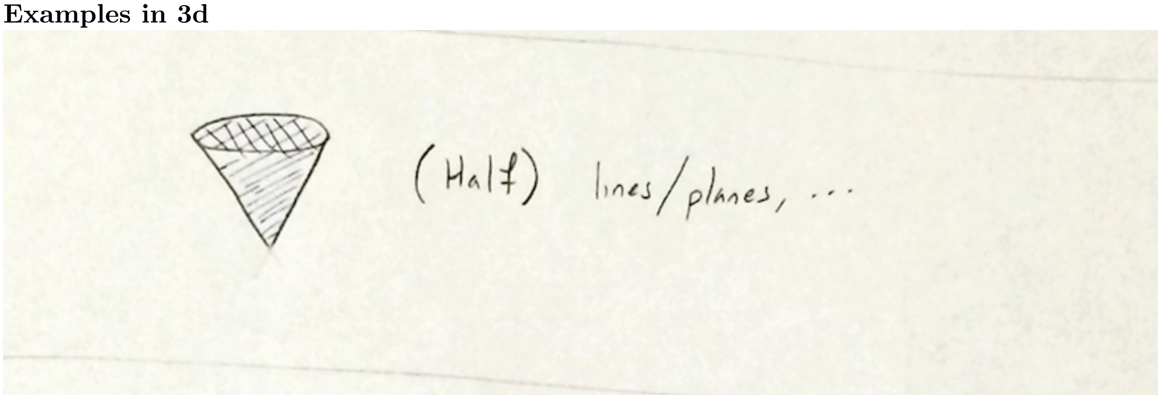
Grasping

Definition $\mathcal{C} \subset \mathbb{R}^n$ is called a *convex cone* for any $x, y \in \mathcal{C}$ and $\alpha, \beta > 0$ s.t. $\alpha x + \beta y \in \mathcal{C}$.

Examples in 2d



Examples in 3d



Lemma $FC \subseteq \mathbb{R}^m$ is a convex cone.

Proof

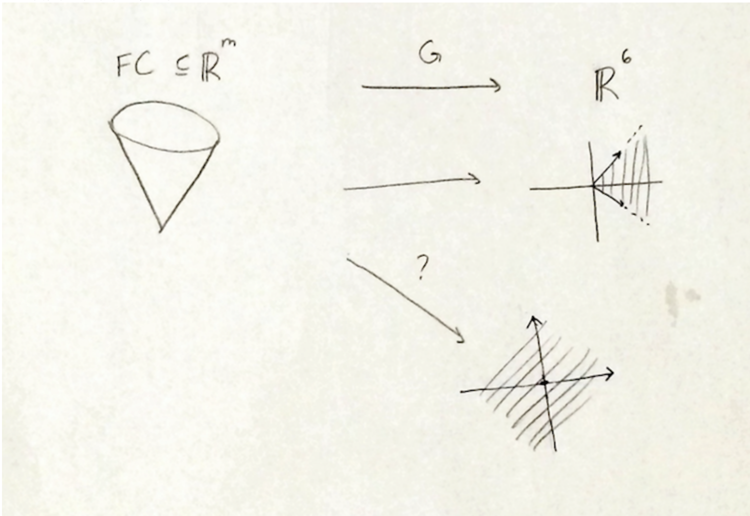
$$\text{Let } f, f' \in FC \text{ and } \alpha_1 f > 0,\ f = (f_1, ..., f_k) \in FC = FC_1 \times ... \times FC_k$$

$$f' = (f'_1, ..., f'_k) \in FC$$

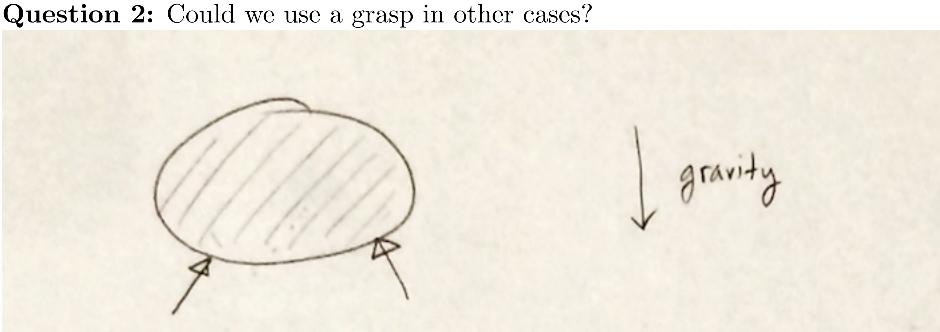
$$\alpha f + \beta f' \in FC \Leftrightarrow \alpha f_i + f'_i \in FC_i,\ \forall i$$

but each FC_i is a convex cone (exercise).

Abstract View



Question 1: Which cone is better?
Question 2: Could we use a grasp in other cases?



Definition
A grasp is a *force-closure* grasp if given any external wrench $F_e \in \mathbb{R}^p$ applied to the object, there exists contact forces $f_c \in FC$ such that

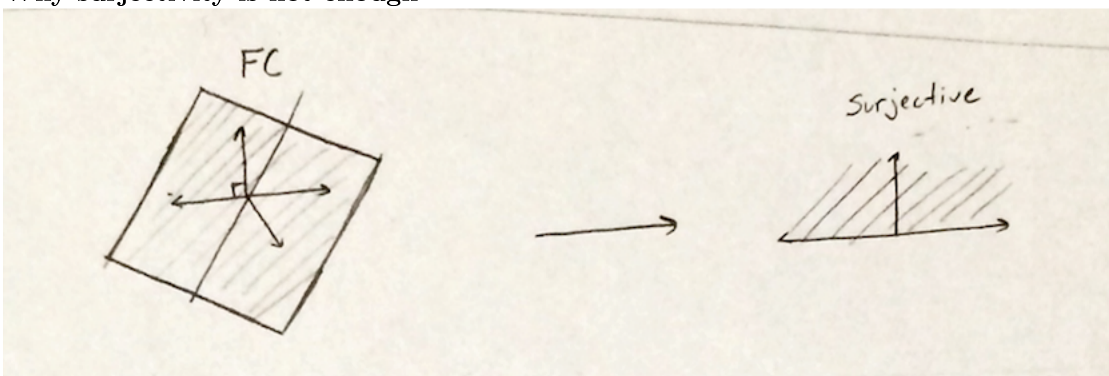
$$Gf_c = -F_e$$

Lemma A grasp is in force-closure if and only if $G(FC) = \mathbb{R}^p$

Definition
If $f_N \in \mathcal{N}(G) \cap FC$ then f_N is an *internal force.* If $f_N \in \mathcal{N}(G)$, then it is called a *strictly internal force.*

Proposition (Necessity of Internal Forces)
A grasp is in force-closure if and only if G is a surjective map and there exists a vector of contact forces $f_N \in \mathcal{N}(G)$ such that $f_N \in \text{int}(FC)$.

Why surjectivity is not enough



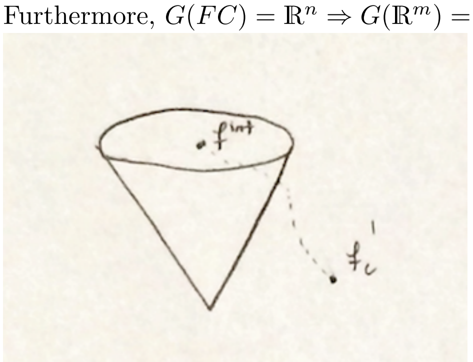
Proof

Suppose G is surjective and $f_N \in \text{int}(FC) \cap \mathcal{N}G$. Let $F_o \in \mathbb{R}^p$. Let f'_c s.t. $F_o = Gf'_c$ (G is surjective!)

$$\text{Note that } \lim_{\alpha\to\infty} \frac{f'_c + \alpha f_N}{\alpha} = \lim_{\alpha\to\infty} \frac{f'_c}{\alpha} + f_N = f_N \in \text{int}(FC)$$

Hence, by the fact that $f_N \in \text{int}(FC)$, $\exists \alpha' > 0$ s.t. $f^{int} = \frac{f'_c + \alpha' f_N}{\alpha'} \in \text{int}(FC)$
Now $f_c = f'_c + \alpha' f_N \in \text{int}(FC)$ and $Gf_c = Gf'_c + \alpha G f_N \in FC$,
so $Gf'_c = F_o$ $f_c \in FC$, so G is in force closure.

On the other hand, suppose the grasp is in force-closure. Pick $f_1 \in \text{int}(FC)$ and let $F_o = Gf_1$.
Then $\exists f_2 \in FC$ where $-F_o = Gf_2$
Then define $f_N = f_1 + f_2$, $Gf_N = 0$ and $f_N \in \text{int}(FC)$ as $f_1 \in$ FC
Furthermore, $G(FC) = \mathbb{R}^n \Rightarrow G(\mathbb{R}^m) = \mathbb{R}^m$



Example Grasp with frictionless point contacts

$$F_o = \begin{bmatrix} R_{c_i} & 0 \\ \hat{p}_{c_i} R_{c_i} & R_{c_i} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} f_{c_i}, \ f_{c_i} \geq 0$$

$$= \begin{bmatrix} R_{c_i} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \\ \hat{p}_{c_i} R_{c_i} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} f_{c_i} \qquad n_{c_i} = R_{c_i} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_{c_i} \\ p_{c_i} \times n_{c_i} \end{bmatrix} f_{c_i}$$

So

$$G = \begin{bmatrix} n_{c_1} & \dots & n_{c_k} \\ p_{c_1} \times n_{c_1} & \dots & p_{c_k} \times n_{c_k} \end{bmatrix} \begin{bmatrix} f_{c_1} \\ \cdot \\ \cdot \\ \cdot \\ f_{c_k} \end{bmatrix} = G f_c$$

$G(FC) = \mathbb{R}^6 \Leftrightarrow$ positive linear combination of columns span $\mathbb{R}^6$.

Similarly in $\mathbb{R}^2$

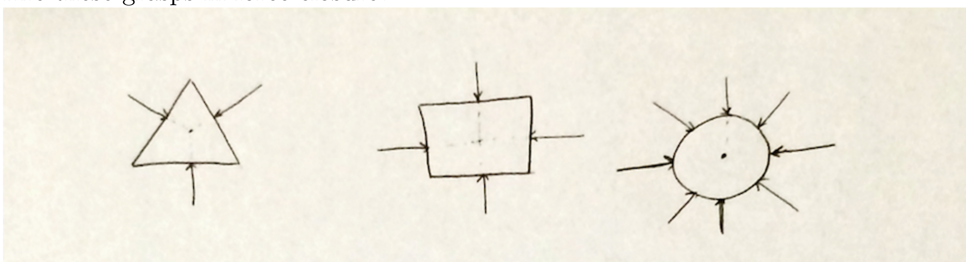
$$F_o = \begin{bmatrix} R_{c_i} & 0 \\ \begin{bmatrix} -p_y & p_x \end{bmatrix} R_{c_i} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} f_{c_i}$$

$$= \begin{bmatrix} n_{c_i} \\ \langle \begin{bmatrix} -p_{iy} & p_{ix} \end{bmatrix}, n_{c_i} \rangle \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} f_{c_i}$$

$$G = \begin{bmatrix} n_{c_1} & \dots & n_{c_k} \\ \langle \begin{bmatrix} -p_{1y} & p_{1x} \end{bmatrix}, n_{c_1} \rangle & \dots & \langle \begin{bmatrix} -p_{ky} & p_{kx} \end{bmatrix}, n_{c_k} \rangle \end{bmatrix}$$

Force Closure

Are these grasps in force closure?



No:

$$G_i = \begin{bmatrix} n_i \\ \langle \begin{bmatrix} -p_y & p_x \end{bmatrix}, n_i \rangle \end{bmatrix}$$

note:

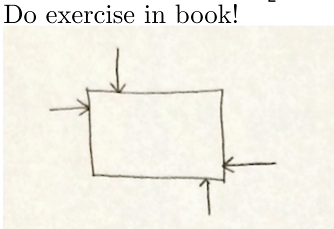
$$\langle \begin{bmatrix} -p_y & p_x \end{bmatrix}, \begin{bmatrix} n_x & n_y \end{bmatrix} \rangle = -p_y n_x + p_x n_y$$

if $\begin{bmatrix} n_x & n_y \end{bmatrix} = \alpha \begin{bmatrix} p_x & p_y \end{bmatrix}$, then $\langle \begin{bmatrix} -p_y & p_x \end{bmatrix}, \begin{bmatrix} n_x & n_y \end{bmatrix} \rangle = 0$

So $G = \begin{bmatrix} x & ... & x \\ x & ... & x \\ 0 & ... & 0 \end{bmatrix}$

Similarly, in 3d, $G_i = \begin{bmatrix} n \\ p \times n \end{bmatrix}$

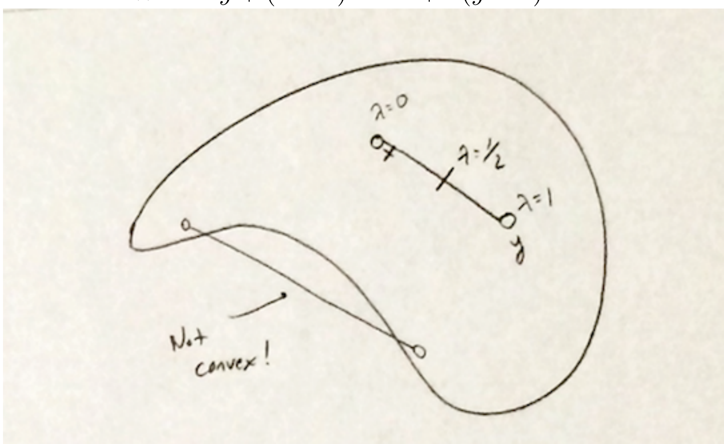
Do exercise in book!



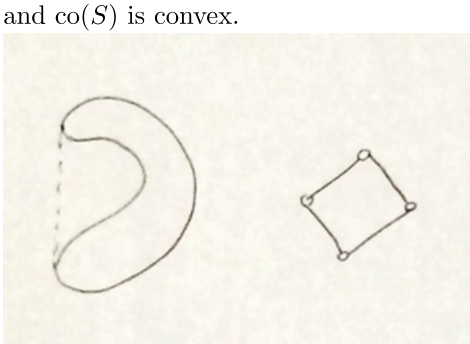
Definition: A set $S \subseteq \mathbb{R}^n$ is *convex* if and only if for all $x, y \in S$

$$\lambda y + (1-\lambda)x \in S \quad \forall \lambda \in [0,1]$$

Intuition Write $\lambda y + (1-\lambda)x = x + \lambda(y - x)$



Definition: For $S \subseteq \mathbb{R}^n$ the convex hull co(S) is the smallest set s.t. $S \subseteq$ co(S) and co(S) is convex.



For a finite set of points, $S = \{v_1, ..., v_k\} \subseteq \mathbb{R}^n$

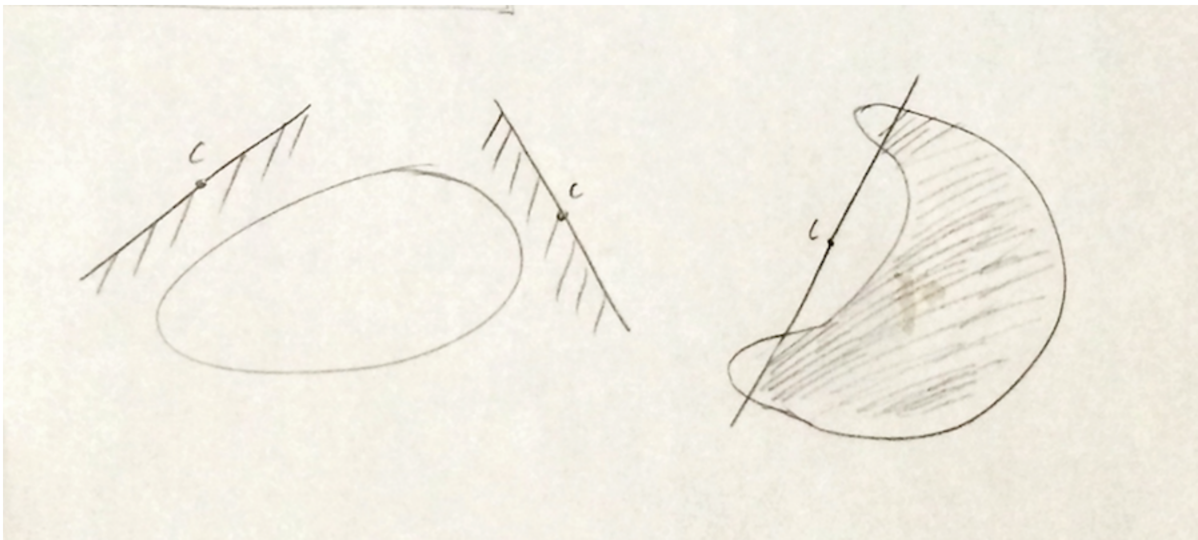
$$\text{co}(S) = \{v = \sum_{i=1}^{k} \alpha_i v_i | \{\alpha_i = 1, \alpha_i \geq 0\}\}$$

Basic Theorem If $S \subseteq \mathbb{R}^n$ is convex and $c \notin S$ then $\exists$ a hyperplane

$$H_v(c) = \{x \in \mathbb{R}^n; v^t(x - c) = 0\}$$

and

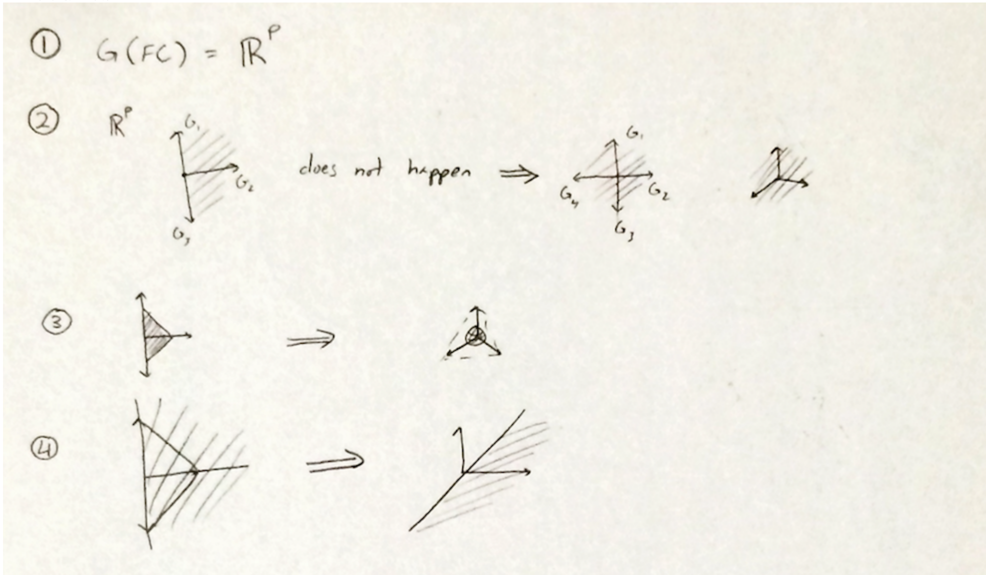
$$S \subseteq \{x | v^t(x - c) > 0\}$$



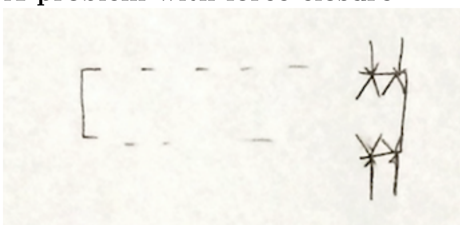
Proposition
Consider a grasp with frictionless point contacts and grasp matrix $G \in \mathbb{R}^{p \times m}, G = (G_1, ..., G_m)$
The following are equivalent:

1. The grasp is in force closure
2. $\{\sum_{i=1}^{m} \alpha_i G_i | \alpha_i > 0\} = \mathbb{R}^p$
3. $\mathrm{co}(\{G_1, ..., G_m\}) \cap \mathbb{B}_r(o) \neq 0$
   for some small $r > 0$ (the convex hull contains a neighborhood of the origin)
4. $\nexists v \in \mathbb{R}^p, v \neq 0$ s.t. $v \cdot G_i \geq 0 \forall i \in \{1, ..., m\}$

Pictures



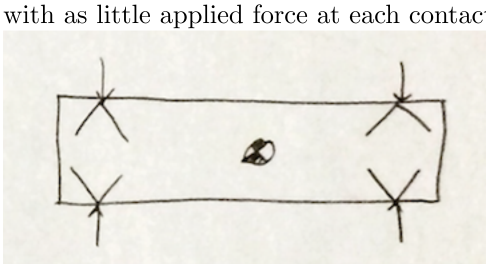
A problem with force closure



Question: How good is a given force closure grasp?
Idea: We want to be able to resist a given wrench F_o.

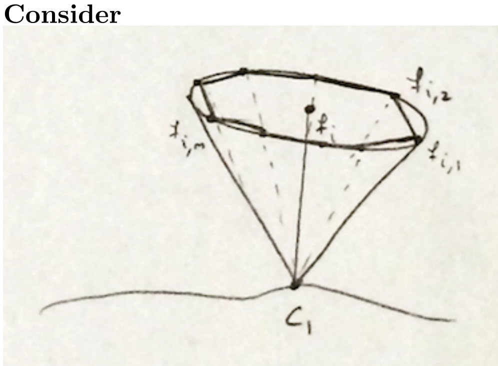
$$F_o = G f_c$$

with as little applied force at each contact as possible.



This leads to the idea of "grasp quality"
We consider "Planning Optimal Grasps" C.Ferrari and J.Canny, ICRA '92

Consider



$$f_i \cong \sum_{j=1}^{m} \alpha_{i,j} f_{i,j}$$

Corresponding wrench with $||f_i^n|| \leq 1$:

$$\begin{aligned} w_i =& G_i f_i = \begin{bmatrix} f_i \\ c_i \times f_i \end{bmatrix} \\ =& \sum_{j=1}^{m} \alpha_{i,j} \begin{bmatrix} f_{i,j} \\ c_i \times f_{i,j} \end{bmatrix} \\ \alpha_{i,j} &\geq 0, \ \sum \alpha_{i,j} \leq 1 \end{aligned}$$

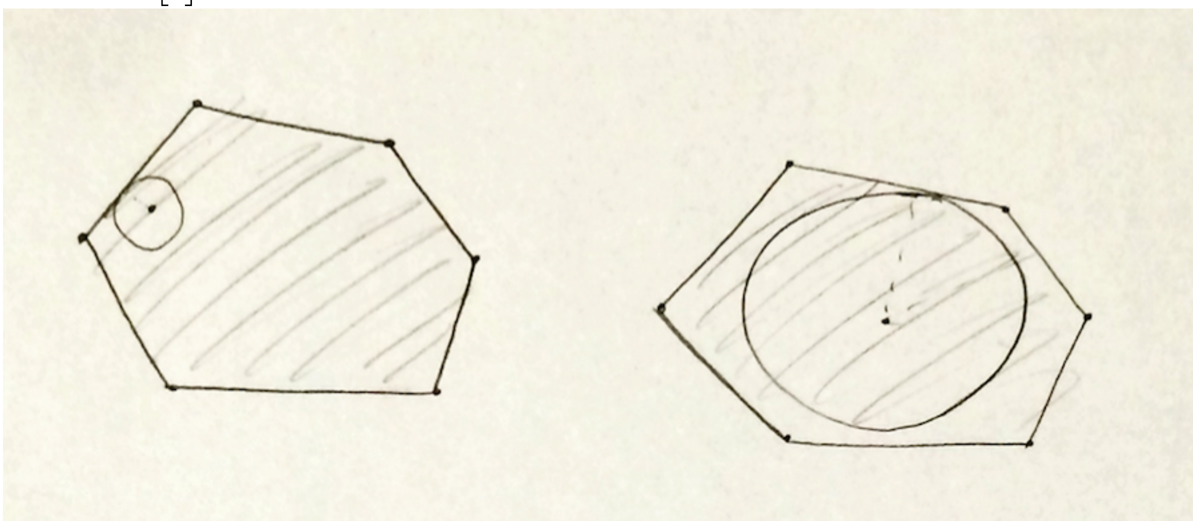
Total wrench for contacts $c_1, ..., c_n$

$$w = \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j} \begin{bmatrix} f_{i,j} \\ c_i \times f_{i,j} \end{bmatrix}$$

$$\alpha_{i,j} \geq 0, \ \sum \alpha_{i,j} \leq 1$$

i.e. $w \in W_{L_1} = \mathrm{co}(W_{i,j} | i \in \{1, ..., n\}, j \in \{1, ..., m\})$

Let $||w|| = || \begin{bmatrix} f \\ \tau \end{bmatrix} = \sqrt{||f||^2 + \lambda||\tau||^2}$



$B_r(o) = \{w \in \mathbb{R}^p | ||w|| \leq r\}$
F&C compute a grasp quality metric in terms of the largest ball in W_{L_1} around the origin.

Issues

Computation relies on discetization

Continuous?

Upper and lower bounds

Gradients and optimization

Friction coefficient!

(a) Robotic Grasping and Contact: A review (Bicchi et al)
(b) Grasp Quality Measures (R Suarez et al)
(c) Classical Grasp Quality Evaluation: New Algorithms and Theory (Pokorny)