Holonomic and Nonholonomic Constraints

Holonomic Constraints

Constraints on the position (configuration) of a system of particles are called *holonomic* constraints.

- Constraints in which time explicitly enters into the constraint equationare called *rheonomic*.
- Constraints in which time is not explicitly present are called *scleronomic*.

Note: Inequalities do not constrain the position in the same way as equality constraints do. Rosenberg classifies inequalities as nonholonomic constraints.

- Particle is constrained to lie on a plane:

$$A\,x_1 + B\,x_2 + C\,x_3 + D = 0$$

- A particle suspended from a string in three dimensional space.

$$(x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - r^2 = 0$$

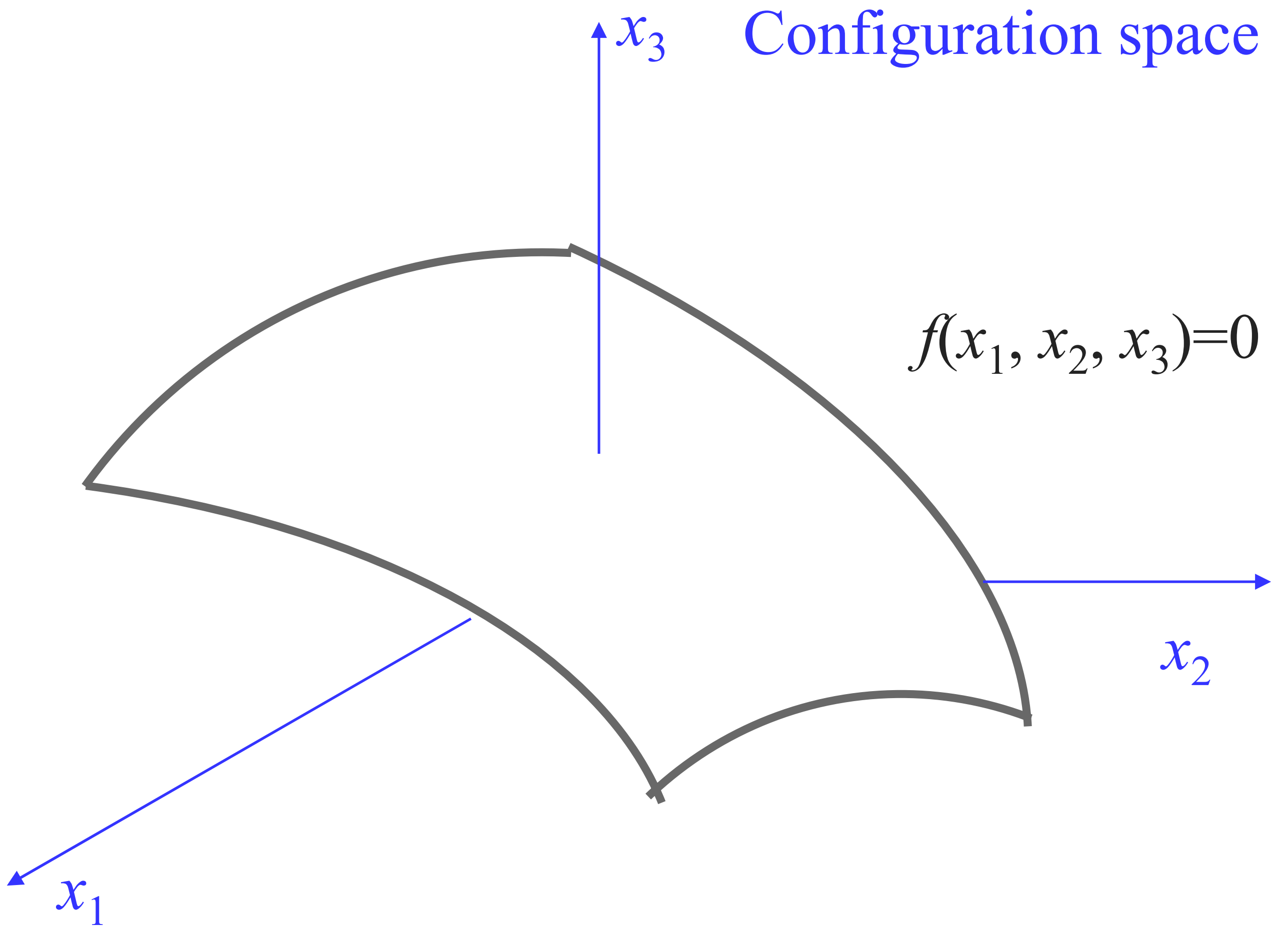
- A particle on spinning platter (carousel)

$$x_1 = a\cos(\omega t + \phi);\ x_2 = a\sin(\omega t + \phi)$$

- A particle constrained to move on a circle in three-dimensional space whose radius changes with time *t*.

$$x_1\ dx_1 + x_2\,dx_2 + \ x_3\,dx_3\ - c^2\,dt = 0$$

Holonomic Constraint

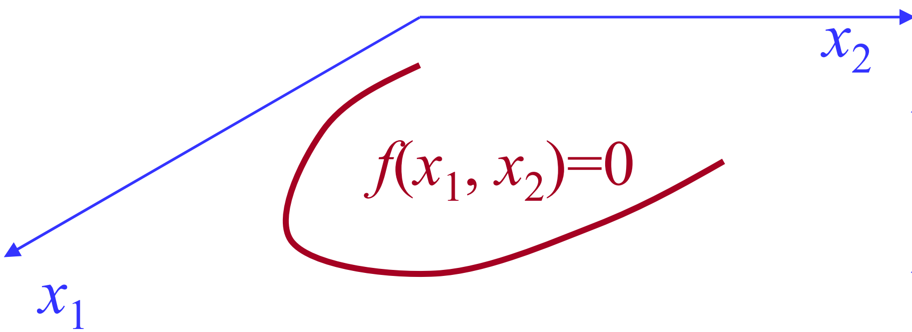


Scleronomic and Rheonomic

Scleronomic

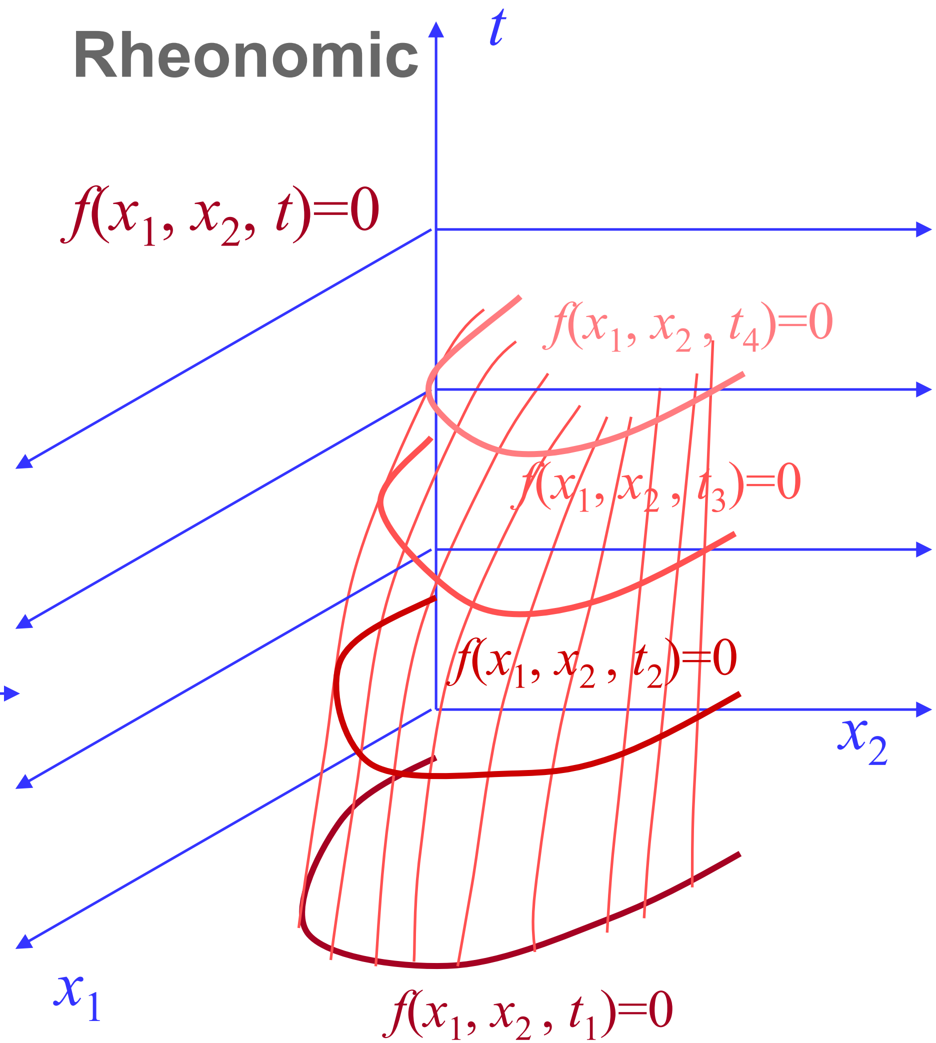
$f(x_1, x_2)=0$

Configuration space



Rheonomic

$f(x_1, x_2, t)=0$



Nonholonomic Constraints

Definition 1

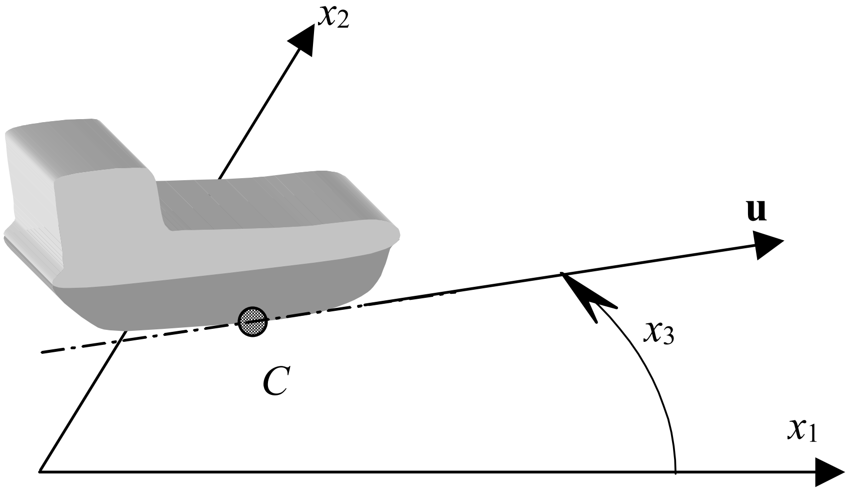
- All constraints that are not holonomic

- A particle constrained to move on a circle in three-dimensional space whose radius changes with time *t*.

$$x_1 \, dx_1 + x_2 \, dx_2 + x_3 \, dx_3 - c^2 \, dt = 0$$

- The *knife-edge constraint*

$$\dot{x}_1 \sin x_3 - \dot{x}_2 \cos x_3 = 0$$



When is a constraint on the motion nonholonomic?

Velocity constraint

$$a_1\dot{x}_1 + a_2\dot{x}_2 + \ldots + a_{n-1}\dot{x}_{n-1} + a_n = 0$$

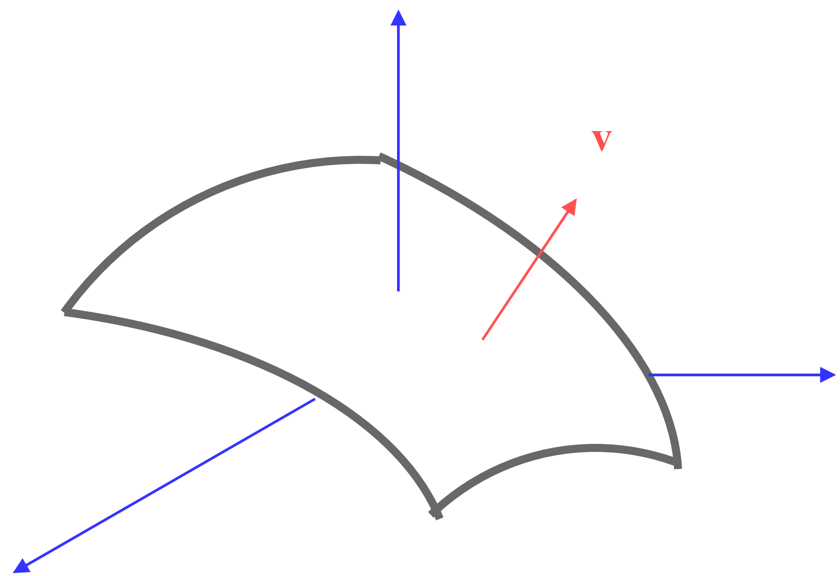
3 dimensional case

$$P\,dx + Q\,dy + R\,dz = 0 \qquad \mathbf{v} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Or constraint on instantaneous motion

$$a_1 dx_1 + a_2 dx_2 + \ldots + a_{n-1} dx_{n-1} + a_n dt = 0$$

Pfaffian Form



Question

Can the above equation can be reduced to the form:

$$f(x_1, x_2, \ldots, x_{n-1}, t) = 0$$

When is a scleronomic constraint on motion in a three-dimensional configuration space nonholonomic?

Velocity constraint

$$P\dot{x} + Q\dot{y} + R\dot{z} = 0$$

Or constraint in the Pfaffian form

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

Question

Can the above equation can be reduced to the form:

$$f(x, y, z) = 0$$

Or,

Can we at least say when the differential form (1) an exact differential?

$$df = P\,dx + Q\,dy + R\,dz$$

- A **sufficient** condition for (1) to be integrable is that the differential form is an exact differential.
- If it is an exact differential, there must exist a function *f*, such that

$$P = \frac{\partial f}{\partial x}, \qquad Q = \frac{\partial f}{\partial y}, \qquad R = \frac{\partial f}{\partial z}$$

- The necessary and sufficient conditions for this to be true is that the first partial derivatives of *P*, *Q*, and *R* with respect to *x*, *y*, and *z* exist, and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}.$$

Recall Stokes Theorem!

When is a scleronomic constraint on motion in a three-dimensional configuration space nonholonomic?

Constraint in the Pfaffian form

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

Question

Can the above equation can be reduced to the form:

$$f(x, y, z) = 0$$

For the constraint to be *integrable*, it is necessary and sufficient that there exist an integrating factor $\alpha(x, y, z)$, such that,

$$\alpha P\,dx + \alpha Q\,dy + \alpha R\,dz = 0 \quad (2)$$

be an exact differential.

- If (2) is an exact differential, there must exist a function g, such that

$$\alpha P = \frac{\partial g}{\partial x}, \qquad \alpha Q = \frac{\partial g}{\partial y}, \qquad \alpha R = \frac{\partial g}{\partial z}$$

- The necessary and sufficient conditions for this to be true is that the first partial derivatives of P, Q, and R with respect to x, y, and z exist, and

$$\frac{\partial(\alpha P)}{\partial y} = \frac{\partial(\alpha Q)}{\partial x},$$

$$\frac{\partial(\alpha P)}{\partial z} = \frac{\partial(\alpha R)}{\partial x},$$

$$\frac{\partial(\alpha R)}{\partial y} = \frac{\partial(\alpha Q)}{\partial z}.$$

When is a scleronomic constraint on motion in a three-dimensional configuration space nonholonomic?

$$\frac{\partial(\alpha P)}{\partial y} = \frac{\partial(\alpha Q)}{\partial x},$$
$$\frac{\partial(\alpha P)}{\partial z} = \frac{\partial(\alpha R)}{\partial x},$$
$$\frac{\partial(\alpha R)}{\partial y} = \frac{\partial(\alpha Q)}{\partial z}.$$

$$\left(\frac{\partial \alpha}{\partial y}\right)P - \left(\frac{\partial \alpha}{\partial x}\right)Q = \alpha\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),$$
$$\left(\frac{\partial \alpha}{\partial z}\right)P - \left(\frac{\partial \alpha}{\partial x}\right)R = \alpha\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right),$$
$$\left(\frac{\partial \alpha}{\partial y}\right)R - \left(\frac{\partial \alpha}{\partial z}\right)Q = \alpha\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right).$$

$$\mathbf{v} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$\nabla \alpha \times \mathbf{v} = -\alpha \nabla \times \mathbf{v}$$

Necessary and sufficient condition for (2) to be holonomic, provided **v** is a well-behaved vector field and

$$\mathbf{v} \cdot \nabla \times \mathbf{v} = 0$$

Examples

1. $\sin x_3\, dx_1 - \cos x_3\, dx_2 = 0$

$$\mathbf{v} = \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}$$

2. $2x_2x_3\, dx_1 + x_1x_3\, dx_2 + x_1x_2\, dx_3 = 0$

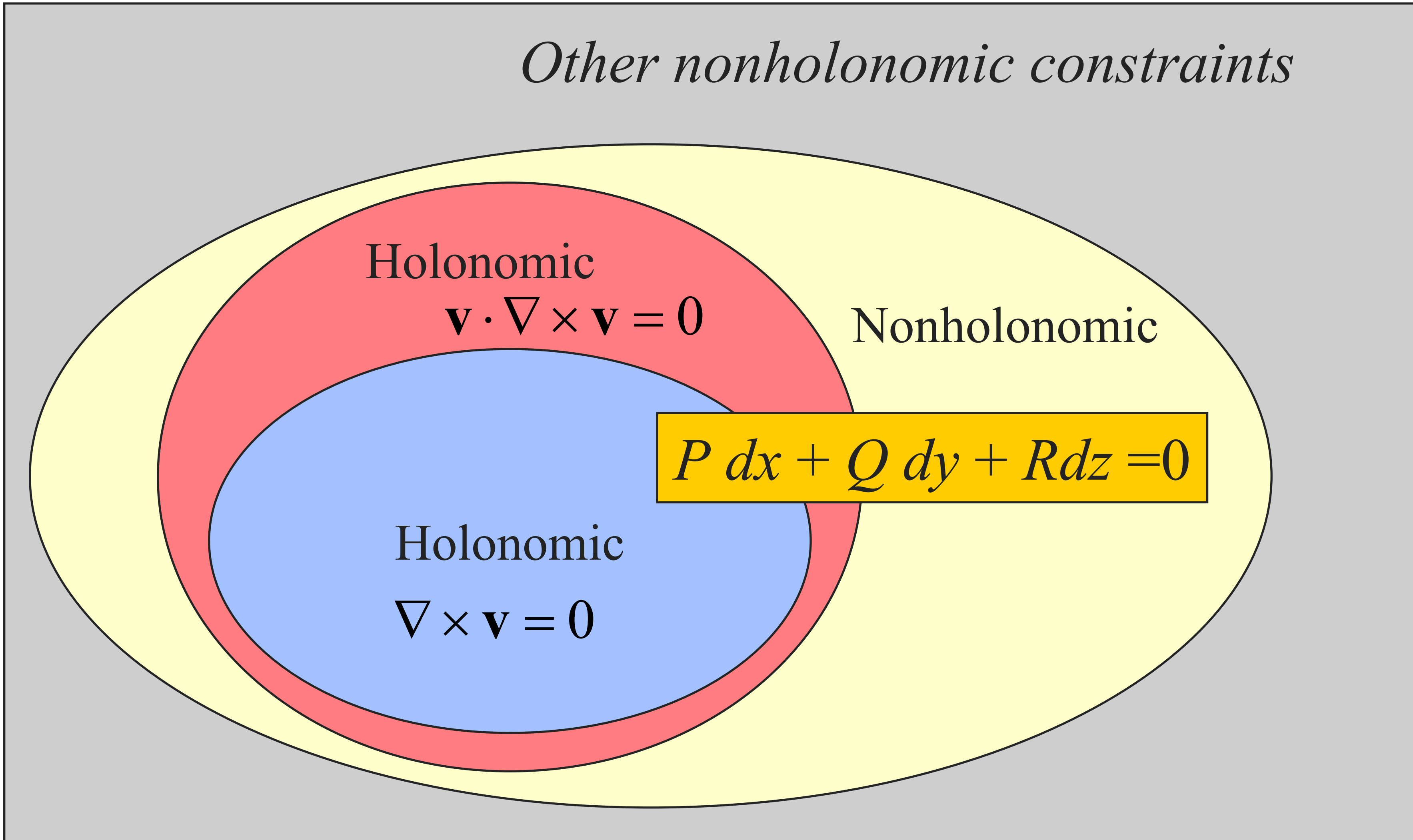
$$x_1\,(2x_2x_3\, dx_1 + x_1x_3\, dx_2 + x_1x_2\, dx_3) = 0$$

$$\mathbf{v} = \begin{bmatrix} 2x_2x_3 \\ x_1x_3 \\ x_1x_2 \end{bmatrix}$$

$$d((x_1)^2\, x_2\, x_3) = 0$$

3. $\dot{x}_1\dot{x}_2 - \dot{x}_3 = 0$

Nonholonomic constraints in 3-D



Extensions: 1. Multiple Constraints

$$\mathrm{d}x_2 - x_3\,\mathrm{d}x_1 = 0$$

and

$$\mathrm{d}x_3 - x_1\,\mathrm{d}x_2 = 0$$

Are the constraint equations non holonomic?

Individually: YES!

Together:

$$\mathrm{d}x_3 - x_1\,\mathrm{d}x_2 = \mathrm{d}x_3 - x_1\,(x_3\,\mathrm{d}x_1) = 0 \qquad x_3 = ke^{\frac{x_1^2}{2}}, \qquad x_2 = \int ke^{\frac{x_1^2}{2}}\,dx_1 + c$$

Extensions 2: Constraints in > 3 generalized speeds

- n dimensional configuration space
- m independent constraints (i=1,..., m)

$$\sum_{j=1}^{n} a_{ij} dx_j - b_i dt = 0$$

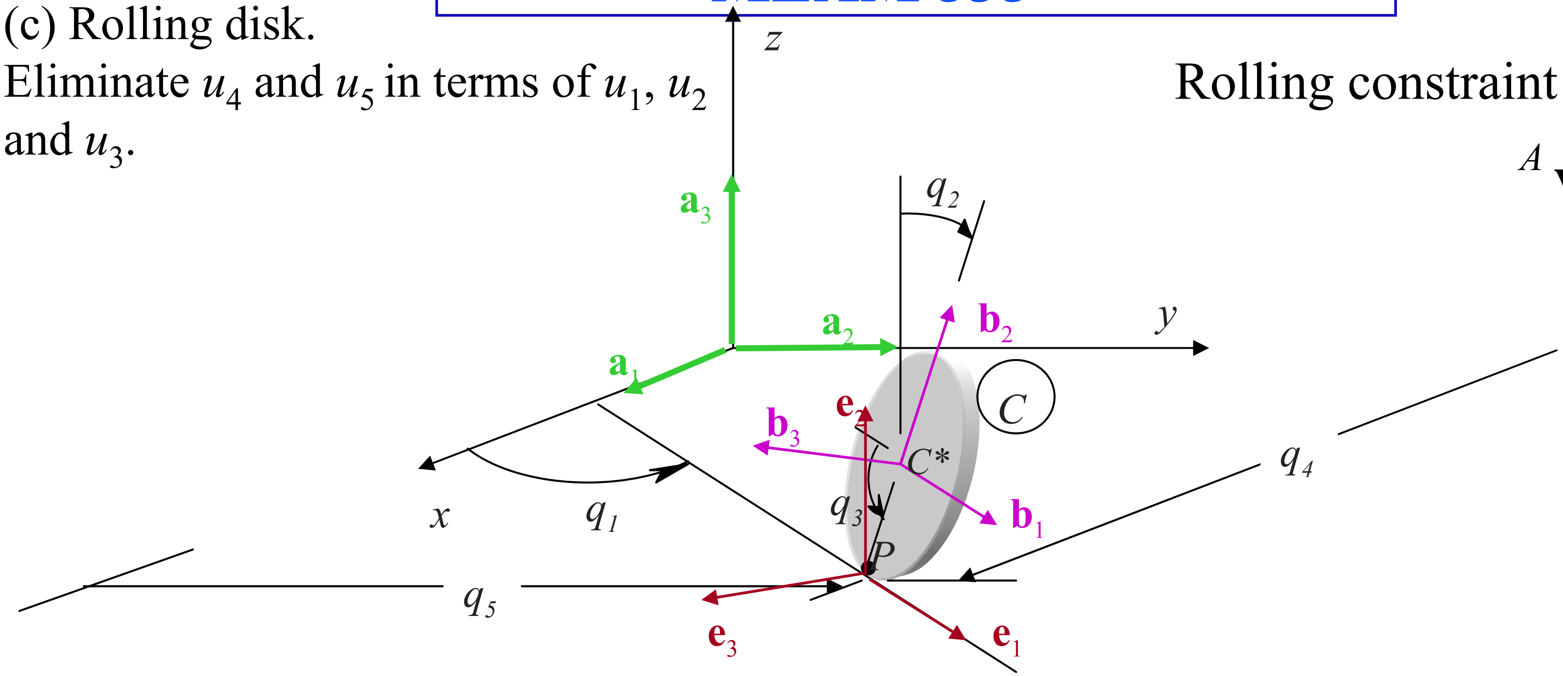
or

$$\sum_{j=1}^{n} a_{ij} u_j - b_i = 0$$

(c) Rolling disk.
Eliminate u_4 and u_5 in terms of u_1, u_2 and u_3.

Rolling constraint

$$^A\mathbf{v}^P = 0$$



$$\begin{aligned} ^A\mathbf{v}^P &= {}^A\mathbf{v}^{C^*} + {}^A\omega^C \times \overrightarrow{C^*P} \\ &= {}^A\mathbf{v}^{C^*} + (u_1\mathbf{b}_1 + u_2\mathbf{b}_2 + u_3\mathbf{b}_3) \times (-R\mathbf{b}_2) \\ &= 0 \end{aligned}$$

$$u_4 \cos q_1 + u_5 \sin q_1 - Ru_2 \tan q_2 + Ru_3 = 0$$

$$-u_4 \sin q_1 \sin q_2 + u_5 \cos q_1 \sin q_2 = 0$$

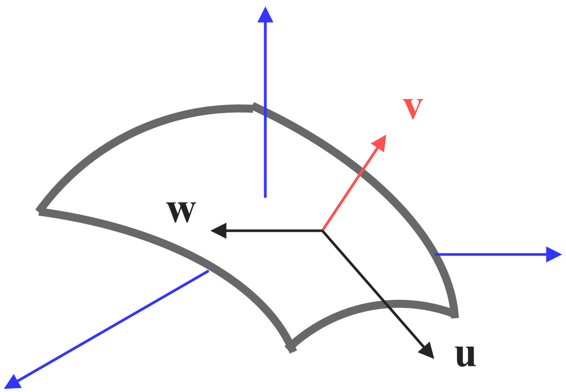
m=2 constraints
n=5 generalized coordinates

Frobenius Theorem: Generalization to *n* dimensons

n dimensional configuration space

m independent constraints (i=1,..., m)

$$\sum_{j=1}^{n} a_{ij} dx_j = 0$$



The necessary and sufficient condition for the existence of m independent equations of the form:

$$f_i(x_1, x_2, \ldots, x_n) = 0, \qquad i=1,\ldots, m.$$

is that the following equations be satisfied:

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial a_{il}}{\partial x_k} - \frac{\partial a_{ik}}{\partial x_l}\right) u_k w_l = 0$$

where u_k and w_l are components of any two n vectors that lie in the null space of the mxn coefficient matrix $\mathbf{A} = [a_{ij}]$:

$$\sum_{j=1}^{n} a_{ij} u_j = 0, \quad \sum_{j=1}^{n} a_{ij} w_j = 0,$$

Generalized Coordinates and Speeds

Holonomic Systems

Number of degrees of freedom of a system in any reference frame

- the minimum number of variables to completely specify the position of every particle in the system in the chosen reference

The variables are called <u>generalized coordinates</u>

There can be no holonomic constraint equations that restrict the values the generalized coordinates can have.

q_1, q_2, ..., q_n denote the generalized coordinates for a system with n degrees of freedom in a reference frame A.

n generalized coordinates specify the position (configuration of the system)

The number of independent speeds is also equal to n

In a system with n degrees of freedom in a reference frame A, there are n scalar quantities, u_1, u_2, ..., u_n (for that reference frame) called <u>generalized speeds</u>. They that are related to the derivatives of the generalized coordinates by :

$$u_i = \sum_{j=1}^{n} Y_{ij}(q_1, q_2, \ldots, t)\dot{q}_j + Z_i(q_1, q_2, \ldots, t)$$

where the nxn matrix $\mathbf{Y} = [Y_{ij}]$ is non singular and $\mathbf{Z}$ is a nx1 vector.

Example 1

Generalized Coordinates

$q_1 , q_2 , q_3 , q_4 , q_5$

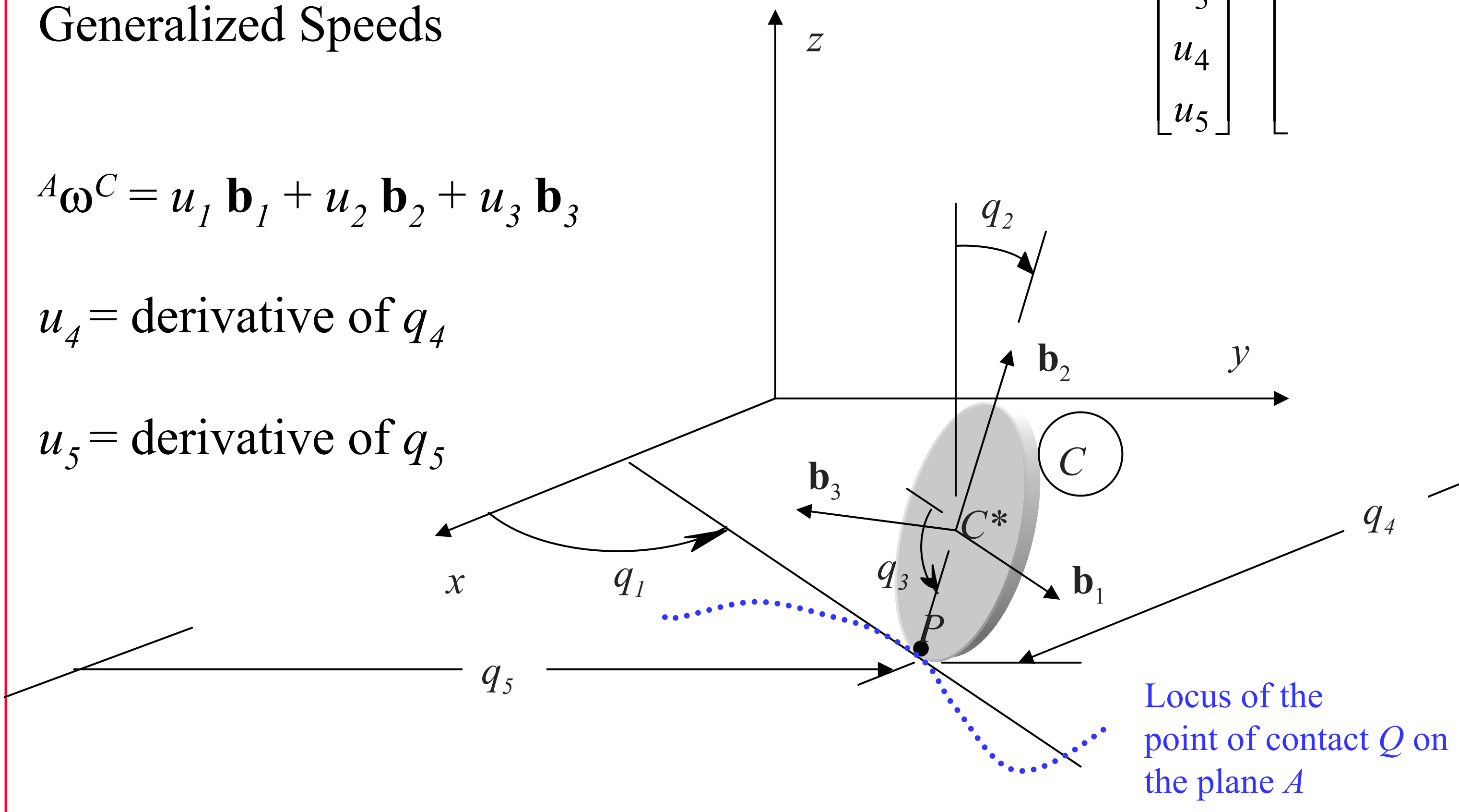
Generalized Speeds

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} & & \\ & \mathbf{Y} & \\ & & \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix}$$

$${}^A\omega^C = u_1\, \mathbf{b}_1 + u_2\, \mathbf{b}_2 + u_3\, \mathbf{b}_3$$

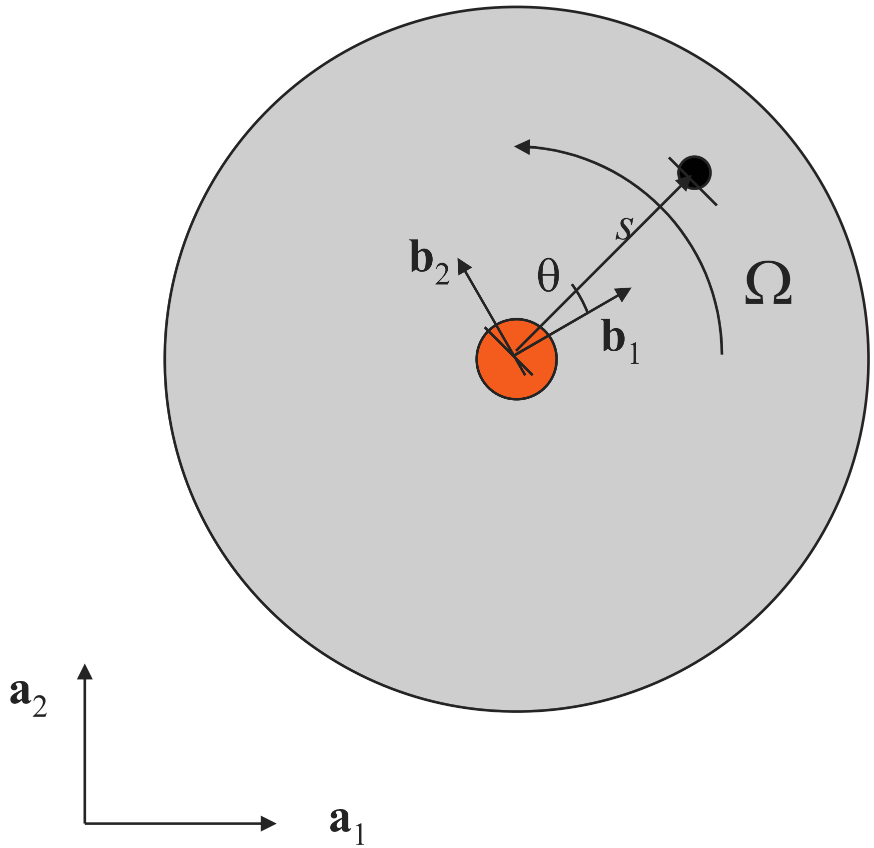
u_4 = derivative of q_4

u_5 = derivative of q_5



Example 2

Bug on the turntable



Generalized coordinates in *A*

- s, θ

Generalized speeds

$$u_1 = \frac{dx}{dt}$$

$$u_2 = \frac{dy}{dt}$$

Generalized speeds and derivatives of generalized coordinates

$$u_i = \sum_{j=1}^{n} Y_{ij}(q_1, q_2, \ldots, t)\dot{q}_j + \boxed{Z_i(q_1, q_2, \ldots, t)}$$

Appears in
rheonomic constraints

Nonholonomic Constraints are Written in Terms of Speeds

m constraints in *n* speeds

$$\sum_{j=1}^{n} C_{ij}(q_1, q_2, \ldots, t) u_j + D_i(q_1, q_2, \ldots, t) = 0$$

m speeds are written in terms of the *n*-*m* (*p*) independent speeds

$$u_i = \sum_{k=1}^{p} A_{ik}(q_1, q_2, \ldots, t) u_k + B_i(q_1, q_2, \ldots, t) = 0$$

Define the *number of degrees of freedom for a nonholonomic system* in a reference frame *A* as *p*, the number of independent speeds that are required to completely specify the velocity of any particle belonging to the system, in the reference frame *A*.

Example 3

Number of degrees of freedom

- $n - m = 2$ degrees of freedom

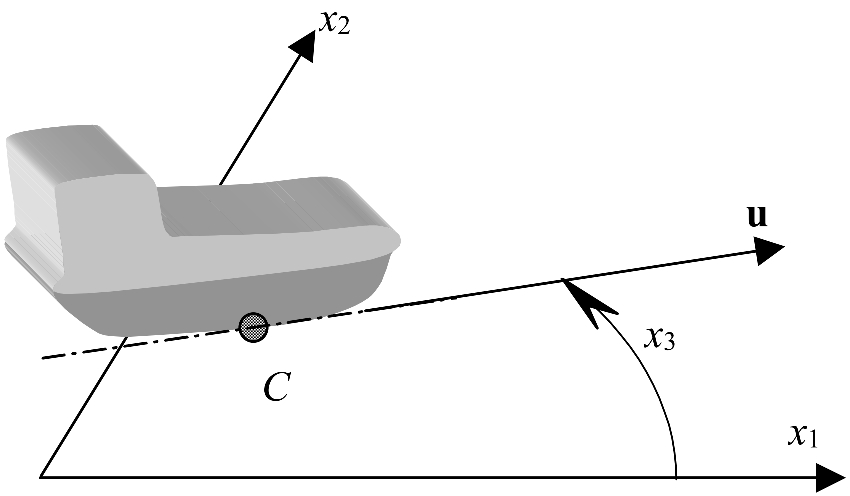
Generalized coordinates

- (x_1, x_2, x_3)

Speeds

- forward velocity along the axis of the skate, v_f
- the speed of rotation about the vertical axis, ω
- and the lateral (skid) velocity in the transverse direction, v_l

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_f \\ \omega \\ v_l \end{bmatrix}$$



$$\mathbf{u} = \mathbf{Y}\dot{\mathbf{q}} + \mathbf{Z}$$

$$\mathbf{Y} = \begin{bmatrix} \cos x_3 & \sin x_3 & 0 \\ 0 & 0 & 1 \\ -\sin x_3 & \cos x_3 & 0 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$