1 Some Observations and Mistakes:

These are some observations inspired on mistakes found on the homework.

- On an arbitrary ring not every principal ideal is prime. Moreover, on an arbitrary ring R given an irreducible element $r \in R$ it is not necessarily true that $(r) \subseteq R$ is a prime ideal. There is a discussion about this in this Stack Exchange Post: https://math.stackexchange.com/questions/1871637/irreducible-but-not-prime-element

- For the last problem some students forgot to use the hypothesis that R was a domain and tried to prove it. One cannot necessarily conclude that if R is a finite k-algebra then R has to be a domain. Indeed, the k-algebra $k[x]/(x^2)$ is a finite k-algebra that is not a domain.

- On problem 14 there was a step that many students did not notice needed justification. In the setup of one of the solutions there was an ideal $I \subseteq \mathcal{O}$ and a maximal ideal $M \subseteq \mathcal{O}$, and the fractional ideal $M^{-1}I$ was considered. This fractional ideal is an ideal of $\mathcal{O}$ if and only if $I \subseteq M$. Indeed, we may multiply the containment $I \subseteq M$ by M^{-1} to get the identity $M^{-1}I \subseteq \mathcal{O}$. Although it is very very simple to prove most proves required in an essential way to make the observation that $M^{-1}I$ is indeed an ideal and not just an arbitrary fractional ideal.

- For the unique factorization part of problem 14 an important lemma was missed often. In a Dedekind domain $\mathcal{O}$ the following holds: if $\mathcal{P}$ and $\mathcal{Q}$ are non-zero prime ideals of $\mathcal{O}$ such that $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{P} = \mathcal{Q}$. Another way of phrasing this is that in a Dedekind domain the only non-zero prime ideals are maximal ideals. Many students only proved that in case that an ideal has more than one factorization, say $J = \prod \mathcal{P}_i = \prod \mathcal{Q}_i$ we would have a containment $\mathcal{P}_i \subseteq \mathcal{Q}_j$, and appeal to a cancelation argument without proving that $\mathcal{P}_i = \mathcal{Q}_j$ which is what one actually needs to make the cancelation argument work.

- Also for problem 14, many many students were using Zorn's lemma incorrectly. The argument was as follows: we considered the set S of ideals of the Dedekind domain $\mathcal{O}$ that did not admit a prime factorization. One assumed S to be non-empty to reach a contradiction. One then had to find (somehow) a maximal element of the set S for the partial order given by containment. In this step two arguments were given one of which was correct and the other incorrect. The correct one would make the observation that by exercise 13 the Dedekind domain is Noetherian and the set of ideals satisfies the ascending chain condition. Any non-empty subset of a partially ordered set that satisfies the ascending chain condition has a maximal element which allowed one to conclude the claim for S. The incorrect argument would appeal to Zorn's lemma to say that S has a maximal element, but this argument ignored a crucial hypothesis required for the use of Zorn's lemma. Indeed, Zorn's lemma says that a partially ordered set for which every ascending chain has an upper bound has a maximal element. What was missing in most of the arguments was to show that every ascending chain in S has an upper bound in S.

- Some students' solutions suggested a confusion between the concepts of a maximal element and greatest element (maximum). In a partially ordered set $(X, \leq)$ we say that an element $x \in X$ is maximal if for all $y \in X$ such that $x \leq y$ we have that $y = x$. A partially ordered set can have many incomparable maximal elements and there is no such thing as "the maximal" element. In a partially ordered set $(X, \leq)$ we say that an element $x \in X$ is the "greatest element" (some people might call it maximum) of X if it is a maximal element of X and every other element of X is comparable to x. We note that not every partially ordered set admits maximal elements, and even if a partially ordered set admits maximal elements it might not have a greatest element.

- There are many equivalent ways of defining Dedekind domains. Exercise 13 and 14 were one direction of proving some of this equivalent definitions. The exercise in the book was most probably designed to be a good exercise if one starts from the definition given in the book, and it is an exercise that leads you to prove part of the theory of Dedekind domains. No points were deducted from using a different definition but I want to encourage you to do the exercises as intended. It is not a very challenging exercise to prove that a Dedekind Domain is Noetherian if your definition of Dedekind Domain includes being Noetherian.

- A student made an interesting observation to solve exercise 14. On a Dedekind domain the following holds: If I is an non-prime ideal then there are non-unital ideals A and B with $AB = I$. This will generally not be true for arbitrary rings, that is, there are rings R with an ideal $I \subseteq R$ such that I is not a prime ideal but it is irreducible in the monoid of ideals. I did not check this carefully but I think that for k a field the ring $k[t^2, t^3]$ and ideal (t^3) should be such an example.

2 Some writing remarks

- Reread your homework before turning it in.
- Explain your solution in a way that someone that does not a priori know the solution can understand it. Often times the only reason that I get to know that you understood the problem is because I know how to solve it. The purpose of writing math is to explain the solution of a problem to someone that does not know how to solve it.
- Try to give more emphasis to the most important or most dificult part of the argument. It is not great practice to explain the easy (or known) stuff very carefully if on the hard part of the argument there will be some Ninja steps.
- There are two relatively opposing things that are very valuable in a written solution, and good writing in math is about finding the balance between the two. These two aspects are how succinct the solution is and how complete it is. Many people end up skipping important steps for the sake of brevity, this makes the text much harder to follow and can even lead to substantial mistakes. On the other hand, it is often the case that a first solution to a problem has unnecesary steps or unnecesarily lengthy reasoning. Rethinking your solutions to try to make the argument shorter can lead you to a better understanding of the problem and can lead to substantial simplifications. The exercise of finding shortcuts in your arguments is very valuable an practicing this is something that I recommend you start to do as early in your career as possible.
- It is never too early to start using Latex. There have been some solutions to problems that I was not able to understand because of poor handwriting.