1 List of common mistakes and some comments

- Polynomials in several variables over a field behave very differently than polynomials in one variable because $k[x_1]$ is a principal ideal domain but $k[x_1, \ldots, x_n]$ for $n > 1$ is not a principal ideal domain. This fact makes things much more complicated for several variables and here I list some consequences that made most of the mistakes in this weeks homework.
  - The zeros of a polynomial $p(x) \in k[x]$ are called roots. We restrict the usage of the word root for polynomials in one variable because the zeroes of polynomials in several variables behave differently.
  - The zero set can be infinite, example: the polynomial $x_1 \in \mathbb{C}[x_1, x_2]$ vanishes in the set $\{(0, a) | a \in \mathbb{C}\}$ which is an infinite set.
  - The "number of zeros" of a polynomial $p(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ only makes sense if the field k is finite, but even in this case the relation of "number of zeros" with the degree of the polynomial is much more complicated.
  - Having non-empty zero set does not prove the polynomial is reducible anymore. Indeed the polynomial $x^2 + y^2 + 1 \in \mathbb{C}[x, y]$ is irreducible but $(i, 0)$ is a zero of this polynomial.
  - If a polynomial in several variables vanishes at a point $(a_1, \ldots, a_n) \in k^n$ it does not imply that you can factor out a linear factor as one can do with roots.
- Over a finite field k with cardinality q, if two polynomials $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ define the same function on k^n it won't be necessarily true that $f = g$. Indeed, the only thing you can say is that they have the same reduction (where by reduction of f we mean the only element of $g \in k[x_1, \ldots, x_n]$ whose degree in each individual variable is smaller than q and such that f and g have the same image in the ring $k[x_1, \ldots, x_n]/(x_1^q - x_1, \ldots, x_n^q - x_n)$).
- When working over a finite field of cardinality q one does not consider congruences $(mod\, q)$. From studying finite fields we know that any finite field has the form $\mathbb{F}_q$ where $q = p^n$ for some prime. But the additive structure of the ring $\mathbb{F}_q$ is p-torsion rather than q-torsion.
- For the last problem, a lot of students only proved that Moebious transformations are automorphisms of $K(x)/K$ but they did not justify why every automorphisms of $K(x)/K$ is given a Moebious transformation.